THE LAMINAR BOUNDARY LAYER, TAKING INTO ACCOUNT RADIATIVE ENERGY TRANSFER

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The integral-differential energy equation is solved, taking into account the viscous dissipation, convection, thermal conductivity, and radiative transfer, without restrictions on the optical thickness of the boundary layer. Comparison of the calculated results with the solution obtained without taking into account the absorption makes it possible to evaluate the relative role of absorption. The influence of radiation on the enthalpy profile and thickness of the boundary layer is investigated.

It is considerably easier to take account of radiation in a boundary layer if we use the approximations of greater or lesser optical thickness. For a gray gas in a state of local thermodynamic equilibrium, these approximations reduce the integral-differential equation system to a purely differential system. A radiating boundary layer has been considered in these approximations by various authors [1-5], with different assumptions regarding the viscosity, thermal conductivity, and convection. Oliver and McFadden [6] investigated a radiating boundary layer at a plate, taking into account the radiative term in the exact energy equation, which is valid for all optical thicknesses. The mass absorption constant was considered to be independent of the frequency, temperature, and gas density. The integral-differential energy equation is also solved in the present article, without any restrictions on the optical thickness of the boundary layer. In contrast to the study by Oliver and McFadden [6], however, the absorption constant is assumed to be the same as in heated air, i.e., depends strongly on temperature and density. The solution of the integral-differential energy equation is compared with the solution obtained in the approximation of small optical thickness, i.e., without taking into account the absorption. Comparison of these solutions enables us to evaluate the relative role of absorption in radiative energy transfer.

We will consider the flow of a radiating, viscous, heat-conductive gas by a plate, with a constant specific heat capacity ratio. Our assumption of a linear relationship between the enthalpy and temperature does not wholly correspond to the real properties of air at high temperatures. However, the inaccuracy due to use of this assumption in the solutions obtained with and without the absorption taken into account is the same, so that comparison of these solutions permits evaluation of the relative role of absorption in radiative energy transfer in the boundary layer.

A "gray" gas is assumed to be in a state of local thermodynamic equilibrium. Taking the radiation into account under the conditions in question leads only to the appearance of an additional term in the energy equation. In the approximation of a small optical thickness, this additional term takes into account the two-dimensionality of the radiation field. In finding the solution with the absorption taken into account, we neglect the radiative energy transfer along the boundary layer. The latter assumption can be made if the change in temperature along the boundary layer over the radiation path length is small. Because of the comparatively slow variation of all the gas parameters in the longitudinal direction, the radiation field can be treated as homogeneous except in the case of large radiation path lengths. In the latter case, the absorption becomes an effect of second order of smallness in comparison with the emission, and the radiation term in the energy equation can be written in the same form as in the approximation of small optical thickness, i.e., with the two-dimensionality of the radiation field taken into account. The error in the solution that takes account of the absorption (resulting from assumption of one-dimensionality of the radiation field) should therefore not have any material influence on the results. Taking the foregoing into account, the equation system for a radiating boundary layer is written in the form:

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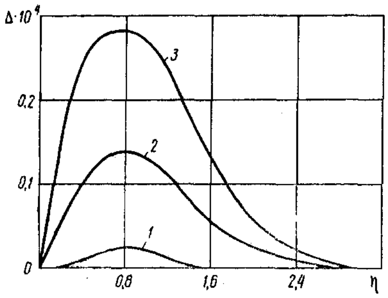

Fig. 1

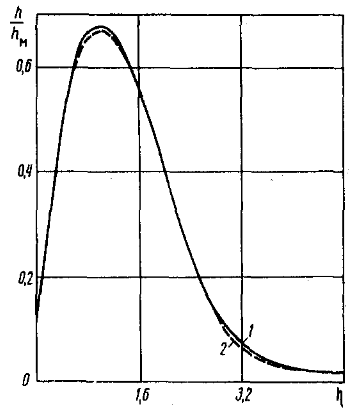

Fig. 2

Fig. 1. Relative increase in enthalpy Δ resulting from absorption as a function of dimensionless coordinate η for $\xi = 0.21$. 1) $M_\infty = 24$; 2) 30; 3) 35.

Fig. 2. Solution of energy equation without absorption taken into account for $M_\infty = 24$ and $\xi = 0.56$. 1) Finite-difference method; 2) expansion method.

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right);$$
$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0; \tag{1}$$
$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{\mathrm{Pr}} \frac{\partial}{\partial y}\left(\mu \frac{\partial h}{\partial y}\right) - \frac{\partial S}{\partial y}.$$

The boundary conditions are

$$\begin{aligned} &\text{when } y = 0 \quad u = v = 0, \quad h = h_w; \\ &\text{when } y \to \infty \quad u \to U_\infty, \quad h \to h_\infty. \end{aligned} \tag{2}$$

An expression for the radiant heat flux S is obtained after integration of the equation for radiation transfer from the plate surface to the upper edge of the boundary layer. The total radiative heat flux for each transverse-coordinate value is found as the difference between the fluxes in the positive and negative directions of the y axis. The external flux is treated as cold (T_∞ = 250°K), nonradiative, and nonabsorptive. External radiation sources are absent. The plate temperature is constant at 2000°K. The radiation produced by the plate itself and its absorption in the boundary layer cannot be taken into account. Under these conditions, we obtain the following expression for the derivative of the radiant heat flux:

$$\frac{\partial S}{\partial y} = 2\varkappa^0 \left\{2\sigma T^4 - \int_0^{\tau_\delta} \sigma T^4(t) E_1(|\tau - t|)\, dt - 2(1-\beta)\left[\int_0^{\tau_\delta} \sigma T^4(t) E_2(t)\, dt\right] E_2(\tau)\right\}. \tag{3}$$

The first term on the right side of Eq. (3) corresponds to the approximation of an optically thin boundary layer, the second takes into account the absorption of the gas in the boundary layer, and the third represents the absorption of the radiation reflected by the wall.

If the dependence of the viscosity on temperature is assumed to be linear, the thermal and dynamic problems are separated. The solution of the dynamic problem is expressed by the usual Blasius function. We will consequently solve only the energy equation. After converting to dimensionless quantities and using the Dorodnitsyn transformation

$$\xi = x'; \quad \eta = \int_0^{y'} \rho' dy'$$

(the apostrophes indicate dimensionless quantities), this equation acquires the form

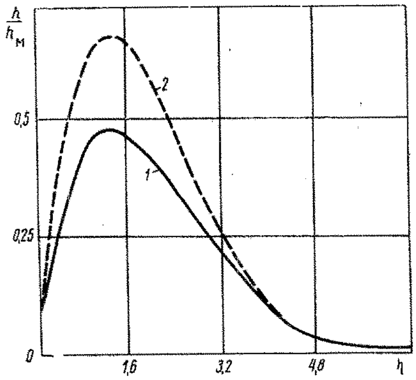

Fig. 3

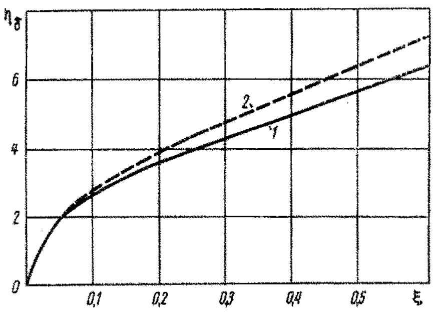

Fig. 4

Fig. 3. Dimensionless enthalpy profiles in boundary layer. 1) With radiation taken into account; 2) without radiation taken into account.

Fig. 4. Change in dimensionless thickness of boundary layer along plate. 1) With radiation taken into account; 2) without radiation taken into account.

$$u\frac{\partial h}{\partial \xi}+\tilde{v}\frac{\partial h}{\partial \eta}=(\gamma-1)M_M^2\left(\frac{\partial u}{\partial \eta}\right)^2+\frac{1}{\mathrm{Pr}}\frac{\partial^2 h}{\partial \eta^2}-2B_0\left\{2h^4-\int\limits_0^{\tau_\delta}h^4(t)E_1(|\tau-t|)\,dt-2(1-\beta)E_2(\tau)\int\limits_0^{\tau_\delta}h^4(t)E_2(t)\,dt\right\}. \quad (4)$$

The boundary conditions are

$$\text{when } \eta=0 \quad h=h_w; \qquad \text{when } \eta\to\infty \quad h\to h_\infty. \quad (5)$$

Here the apostrophes on the dimensionless variables are omitted. The scales for determination of the dimensionless quantities were selected as follows:

$$x_M=L=1\ \text{m};\quad \rho_M=\rho_\infty;$$

$$y_M=\frac{L}{\sqrt{\mathrm{Re}_M}};\quad h_M=0.3h_0\ \text{for}\ M_\infty=24,\ 30;$$

$$u_M=U_\infty;\quad h_M=0.2h_0\ \text{for}\ M_\infty=35.$$

$$v_M=\frac{U_\infty}{\sqrt{\mathrm{Re}_M}}.$$

The velocity profile necessary for solution of Eq. (4) was taken from the usual self-modeling solution without taking account of the radiation.

The radiation properties of the gas were assumed to be the same as in heated air. The volumetric absorption constant was approximated from the following formula:

$$\varkappa^0=A\rho^m T^n. \quad (6)$$

In Eq. (6), the density is measured in g/cm^3 and the temperature in deg K. Traugott [7] gives the dependence of the "gray" absorption constant of air on temperature with different densities. Taking these data into account, the constants in Eq. (6) have the following values:

$$A=0.9\cdot 10^{-19};\quad m=1.5;\quad n=6.$$

In order to simplify evaluation of the integral exponents on the right side of Eq. (4), they can be replaced with sufficient accuracy by the ordinary exponents found from the formula

$$E_2(\tau)=0.813\exp(-1.562\tau). \quad (7)$$

The integral-differential equation (4) is solved with the boundary conditions in Eq. (5) by the finite-difference method proposed by Brailovskaya and Chudov [8]. Integration was carried out from $\xi = 0.01$ with an interval of 0.005 for the longitudinal coordinate and 0.08 for the transverse coordinate. The distribution of the gas parameters in the initial cross-section $\xi = 0.01$ was taken from the well-known self-modeling solution for a nonradiating gas.

The same method was used to find a solution to the simplified energy equation, retaining only the first factor in the radiation term. This corresponds to use of the approximation of small optical thickness, where the absorption is not taken into account.

Calculations were made for Pr = 0.7, $\rho_\infty = 6 \cdot 10^{-3} \rho_0$, $T_\infty = 250°K$, and $M_\infty = 24$, 30, and 35. At higher values of M_∞, the dependence of the absorption constant on temperature and density is no longer described by the formula given above [Eq. (6)] with the values of A, m, and n chosen. Figure 1 shows the increase in enthalpy resulting from absorption as a function of η for one value of ξ and different M_∞. It can be seen that the absorption has almost no effect on the enthalpy in the boundary layer over the temperature range investigated (up to 16,000°K; $M_\infty = 35$) with the initial conditions in question, although its influence increases with rising M_∞. For the conditions under considerations, the integral-differential equation (4) can therefore be replaced by the simplified, purely differential equation obtained from Eq. (4) by discarding the integral terms for the absorption. In this case, the initial assumption of one-dimensionality of the radiation field loses all significance.

A method has been proposed for solution of the simplified energy equation that reduces to solution of several successive ordinary differential equations. The enthalpy is written in the form of a series for powers of the longitudinal coordinate ξ. The coefficients of the series are the unknown functions ζ (self-modeling variables in the solution for a nonradiating gas). Substitution of the series into Eq. (4), taking into account only the first factor in the radiation term, and analysis of the coefficients and powers yield the following expression for the enthalpy:

$$h(\xi, \eta) = \theta_0(\zeta) + (\varepsilon\xi)\theta_1(\zeta) + (\varepsilon\xi)^2\theta_2(\zeta) + (\varepsilon\xi)^3\theta_3(\zeta) + \ldots \tag{8}$$

In seeking θ_0, θ_1, θ_2, θ_3, etc., we obtain the ordinary differential equations

$$\frac{\theta_0''}{Pr} + \theta_0' f_0 + \frac{(\gamma - 1) M_M^2}{4} f_0'' = 0, \tag{9}$$

the boundary conditions

$$\theta_0(0) = h_w; \quad \theta_0(\infty) = h_\infty; \tag{9'}$$

$$\frac{\theta_1''}{Pr} + \theta_1' f_0 - 2\theta_1 f_0' = 4\theta_0^{19/2}, \tag{10}$$

the boundary conditions

$$\theta_1(0) = \theta_1(\infty) = 0; \tag{10'}$$

$$\frac{\theta_2''}{Pr} + \theta_2' f_0 - 4\theta_2 f_0' = 38\theta_0^{17/2}\theta_1, \tag{11}$$

the boundary conditions

$$\theta_2(0) = \theta_2(\infty) = 0; \tag{11'}$$

$$\frac{\theta_3''}{Pr} + \theta_3' f_0 - 6\theta_3 f_0' = 38\theta_0^{17/2}\theta_2, \tag{12}$$

the boundary conditions

$$\theta_3(0) = \theta_3(\infty) = 0 \tag{12'}$$

etc.

The function $\theta_0(\zeta)$ is the self-modeling solution for a nonradiating gas. Equations (10), (11), and (12) with boundary conditions (10'), (11'), and (12') were solved by the elimination method. The series in Eq. (8), which is the solution without the absorption taken into account, converges in the region close to the leading edge of the plate at not overly large values of ε, which depends on M_∞. The region of practical convergence decreases rapidly as M_∞ increases. Figure 2 compares the solution of the simplified energy equation obtained by the finite-difference method with the solution in series form, retaining the first four terms together with $\theta_0(\zeta)$. The comparison is made for $M_\infty = 24$ and $\xi = 0.56$. It can be seen that the greatest difference in the solutions occurs where the derivative $dh/d\eta$ changes abruptly.

Figure 3 shows the enthalpy profiles for a nonradiating gas and with the radiation taken into account. As can be seen from this figure, radiation substantially reduces the enthalpy in the boundary layer and makes the profile steeper. The influence of radiation is greatest in the portion of the boundary layer where the temperature is maximal and the temperature gradient is not overly large. Under the conditions in question, radiation reduces the thickness of the thermal boundary layer. Figure 4 shows the change in boundary-layer thickness as a function of the longitudinal coordinate for $M_\infty = 30$ with and without the radiation taken into account.

NOTATION

γ	is the specific heat ratio;
ρ	is the density;
ρ_0	is the density of air under normal conditions;
x	is the longitudinal coordinate;
y	is the transverse coordinate;
u, v	are the velocity projections on x and y axes;
μ	is the viscosity;
h	is the enthalpy;
Pr	is the Prandlt number;
S	is the radiative heat flux;
U_∞	is the velocity of incoming flow;
β	is the plate thickness;
$\varkappa^0$	is the volumetric constant with forced emission taken into account;
σ	is the Stefan–Boltzmann constant;
T	is the temperature;
M	is the Mach number;
ξ, η	are the Dorodnitsyn variables;
h_0	is the stagnation enthalpy;
Re	is the Reynolds number;
L	is the characteristic linear dimension;
$\varkappa = \varkappa^0/\rho$	is the mass absorption constant;
$\zeta = \eta/2\sqrt{\xi}$	is the self-modeling variable in solution for nonradiating gas;
$f_0(\zeta)$	is the Blasius function;
$\tau = \int_0^y \varkappa^0 dy$	is the optical coordinate;
E_1, E_2	are the integral exponents;
$E_n(\tau) = \int_1^\infty w^{-n} \exp(-w\tau) dw$	
$B_0 = (\sigma T_M^4 \varkappa_M^0 L/\rho_M u_M h_M)$	is the dimensionless parameter characterizing ratio of radiant energy flux to hydrodynamic flux;
$\varepsilon = 4B_0$	
$\tilde{v} = u\, \partial\eta/\partial x + \rho v$	
$\Delta = (h_{abs} - h_{no\,abs})/h_m)$	is the relative increase in enthalpy resulting from absorption.

Subscripts

w	quantities at plate surface;
δ	quantities at outer edge of boundary layer;
∞	parameters of incoming flow;
M	scale values.

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